Problema 1

Fie $a, b, c \in [0,1]$ cu proprietatea $ab+bc+ca=1$.Demonstrați ca $a^2+b^2+c^2 \leq 2$.Precizati situatiile cand are loc egalitatea.

G.M. nr.10/2011

Solutie:

Demonstram ca daca x,y,z$\in$[0, 1] atunci

x+y+z-xy-xz-yz$\leq$1…………………………………………………….1p

Formam functia f:[0, 1]$\rightarrow$R, f(x)=x(1-y-z)+y+z-yz care este functie de gradul I daca 1-y-z$\neq$0.

In concluzie maximul ei se atinge in f(0) sau f(1)……………………..1p

f(0)=y+z-yz=1-(1-y)(1-z)$\leq$1.

f(1)=1-yz$\leq$1$\rightarrow$f(x)$\leq$1…………………………………………………1p

Lund x=a, y=b, z=c obtinem ab+bc+ca=1$\Rightarrow$

a+b+c-1$\leq$1$\Rightarrow$a+b+c$\leq$2………………………………………………1p

Deoarece a, b, c $\in$ [0, 1] $\Rightarrow a^2+b^2+c^2 \leq a+b+c \leq 2$

Prima inegalitaqte devine egalitate $\Leftrightarrow$ a, b, c$\in\{0,1\}$.

Cum stim ca ab+bc+ca=1 in aceasta situatie doua numere coincide cu 1 iar al treilea este 0. Aceasta situatie verifica si a+b+c=2(a II-a egalitate) ……………………………………………………………………2p

Problema 2

Pentru $a\in\mathbb{N}$ notam $a\mathbb{N}=\{an \mid n\in\mathbb{N}\}$. Fie $a, b, c, d\in\mathbb{N}$. Sa se arate ca urmatoarele afirmatii sunt echivalente: *i)* $a\mathbb{N}\setminus b\mathbb{N}\subseteq c\mathbb{N}\setminus d\mathbb{N}$; *ii))* $b|a$ sau $(c|a$ *si* $[a,b] \mid [a,d])$.

Cand are loc egalitatea?

Marin Tolosi si Cosmin Nitu

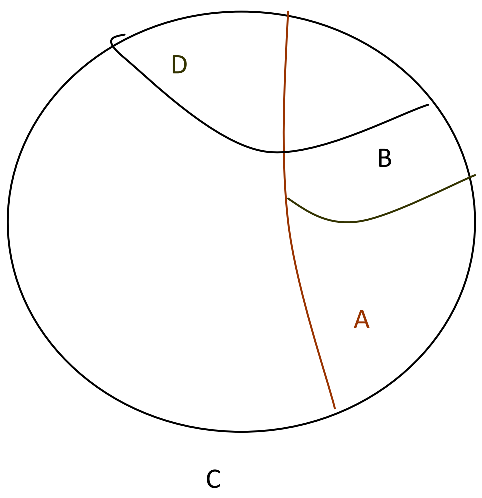
Cand are loc egalitatea?

Marin Tolosi si Cosmin Nitu(in legatura cu problema C.O. 5139 din G.M. nr. 7-8-9/2010)

Solutie.

Lema. Fie A,B, C, D multimi, astfel incat $A\subset C$, $B\subset A$ si $D\subset C$. Atunci $A\setminus B\subset C\setminus D$ daca si numai daca $A\cap D\subset B$.

Demonstrati a lemei. Se poate observa usor acest lucru din urmatoarea diagrama:



"=>" Daca ar exista $b\in(A\cap D)\setminus B$, atunci $b\in A\setminus B$, dar $b\notin C\setminus D$, contradictie.

"<=" Fie $x\in A\setminus B$. Daca $x\in D$, cum $A\cap D\subset B$, rezulta $x\in B$, contradictie.

Asadar, $x \notin D$, deci $x \in C \setminus D$ 2p

Trecem acum la demonstratia propriu-zisa. Vom arata ca $a\square \setminus b\square \subset c\square \setminus d\square$ si, cum $m\square \setminus n\square = m\square \setminus n\square \ \cup \ m\square_{-} \setminus n\square_{-}$, rezulta cerinta problemei. Avantajul este ca pentru $n\square$ avem proprietati suplimentare.

Avem $a\square \setminus b\square = \varnothing$ <=> b|a. Acest caz este trivial. (1)

In caz contrar, $a \in a\square \setminus b\square \subset c\square \setminus d\square$ => $a \in c\square$, deci c | a. (2)

Tinand cont de proprietatile multimilor, avem:

$$a\square \setminus b\square \subset c\square \setminus d\square \Leftrightarrow a\square \setminus [a,b]\square \subset c\square \setminus [c,d]\square$$

Alegem $A = a\square$, $B = [a,b]\square$, $C = c\square$, $D = [c,d]\square$ si aplicam lema, obtinem

$$A \setminus B \subset C \setminus D \Leftrightarrow A \cap D \subset B$$

adica:

$a\square \setminus b\square \subset c\square \setminus d\square \Leftrightarrow a\square \cap [c,d]\square \subset [a,b]\square$ 3p

Dar, $a\square \cap [c,d]\square = [a,c,d]\square = [a,d]\square$ (deoarece c | a), de unde $[a,d]\square \subset [a,b]\square$, ceea ce inseamna $[a,b] \mid [a,d]$. (3)

Tinand cont de (1), (2) si (3), echivalenta este demonstrata.............................2p

Corolar. $a\square \setminus b\square = c\square \setminus d\square \Leftrightarrow$ (b|a si d|c) sau (a=c si [a,b] = [a,d])

Problema 3

Aflati numerele reale *x,y ,z* stiind ca $-2 \leq x \leq y \leq z$, $x+y+z=\frac{2}{3}$

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{3}{8}$$

Cristinel Mortici

Solutie: Avem

$\frac{1}{x^2}-\left(\frac{1}{x}+\frac{3}{16}-\frac{9}{32}x\right)=\frac{1}{32x^2}(3x-4)^2(x+2)$ (1)…………………..2p

deci au loc inegalitatile de tipul

$\frac{1}{x^2} \geq \frac{1}{x}+\frac{3}{16}-\frac{9}{32}x$………………………………………………………1p

Prin adunare,

$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} \geq \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{9}{16}-\frac{9}{32}(x+y+z)$………………………2p

$=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{3}{8}$.

Cum are loc egalitatea, trebuie egalitati in (1), deci x, y,z$\in\{-2,\frac{4}{3}\}$…….1p

$x=-2$, $y=\frac{4}{3}$, $z=\frac{4}{3}$………………………………………………………...1p

Problema 4

Fie un triunghi *ABC* si punctele *M∈(AB), N∈(BC), (CM)∩(AN)={P}* astfel incat $\frac{MA}{MB}=\frac{\alpha}{\beta}\ si\ \frac{NB}{NC}=\frac{\beta}{\gamma}$.

Demonstrati ca: $\frac{1}{\alpha}\overrightarrow{PA}+\frac{1}{\beta}\overrightarrow{PB}+\frac{1}{\gamma}\overrightarrow{PC}=\vec{0}$.

Solutie:

Lema;

Daca ABC este un triunghi si punctele M∈(AB), N∈(AC) si (CM)∩(BN)={O}, iar $\frac{MA}{MB}=u, \frac{NA}{NC}=v$ atunci

$\overrightarrow{OA}+u\overrightarrow{OB}+v\overrightarrow{OC}=\vec{o}$ (sau alta varianta de folosita)…………………………5p

Aplicand aceasta lema in cazul problemei ⇒ $\overrightarrow{PB}+\frac{\beta}{\alpha}\overrightarrow{PA}+\frac{\beta}{\gamma}\overrightarrow{PC}=\vec{O}$, impartind cu β⇒

$\frac{1}{\alpha}\overrightarrow{PA}+\frac{1}{\beta}\overrightarrow{PB}+\frac{1}{\gamma}\overrightarrow{PC}=\vec{O}$ ……………………………………………………2p

Figura este realizata pentru lema.